

Nonexistence results for compressible non-Newtonian fluid with magnetic effects in the whole space

Olga Rozanova

*Department of Differential Equations & Mechanics and Mathematics Faculty,
Moscow State University, Moscow, 119992, Russia*

Abstract

We consider a generalization of the compressible barotropic Navier-Stokes equations to the case of non-Newtonian fluid in the whole space. The viscosity tensor is assumed to be coercive with an exponent $q > 1$. We prove that if the total mass and momentum of the system are conserved, then one can find a constant $q_0 > 1$ depending on the dimension of space n and the heat ratio γ such that for $q \in [q_0, n)$ there exists no global in time smooth solution to the Cauchy problem. We prove also an analogous result for solutions to equations of magnetohydrodynamic non-Newtonian fluid in 3D space.

Key words: compressible non-Newtonian fluid, classical solution, loss of smoothness

1991 MSC: 53Q30

In the present paper we study a system of equations for the velocity $u(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and density $\rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}}_+$ describing the motion of viscous compressible fluid without taking into account of heat phenomena in the n - dimensional space. The system is written as

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0, \quad (1)$$

$$\partial_t(\rho u) + \operatorname{Div}_x(\rho u \otimes u) = \operatorname{Div}_x \mathbb{S}, \quad (2)$$

where \mathbb{S} is the stress tensor. We denote Div and div the divergency of tensor and vector, respectively. The Stokes axioms of the movement of continuum [1]

¹ Supported by DFG 436 RUS 113/823/0-1 and the Ministry of Education of the Russian Federation, project 2.1.1/1399

Email address: rozanova@mech.math.msu.su (Olga Rozanova).

imply the following representation of the stress tensor:

$$\mathbb{S} = \sum_{k=0}^{n-1} \alpha_k(\rho, J_1(\mathbb{D}), \dots, J_n(\mathbb{D})) \mathbb{D}^k, \quad (3)$$

where

$$\mathbb{D} = (D_{ij}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the shear rate tensor and $J_m(\mathbb{D})$, $m = 1, \dots, n$, are its invariants, α_k , $k = 0, \dots, n-1$, are scalar functions.

In particular, for the classical model of Newtonian fluid the stress tensor is given by

$$\mathbb{S} = (\lambda \operatorname{div} u - p(\rho)) \mathbb{I} + 2\mu \mathbb{D}, \quad (4)$$

with constant coefficients of viscosity λ , μ ($\mu > 0$, $\lambda + \frac{2}{n}\mu > 0$) and a given nonnegative function of pressure $p(\rho) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$.

In what follows we assume

$$\mathbb{S} = -p(\rho)\mathbb{I} + \mathbb{P}(\rho, \mathbb{D}), \quad \mathbb{P}(\rho, \mathbb{D}) = \beta_0(\rho, \operatorname{div} u)\mathbb{I} + \beta(\rho, |\mathbb{D}|)\mathbb{D}, \quad (6)$$

where \mathbb{I} is the identity matrix, β_0 and β are smooth functions bounded at zero, $|\mathbb{D}|^2 = \mathbb{D} : \mathbb{D}$,

$$\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^n \mathbb{A}_{ij} \mathbb{B}_{ij}$$

for all tensors \mathbb{A} and \mathbb{B} of rank two. The pressure satisfies the state equation

$$p = A\rho^\gamma, \quad \gamma = \operatorname{const} > 1, \quad A = \operatorname{const} > 0. \quad (7)$$

Moreover, we suppose that \mathbb{P} obeys the integral coercivity condition, i.e.

$$\int_{\mathbb{R}^n} \mathbb{P}(g, \mathbb{B}) : \mathbb{B} \, dx \geq \nu \int_{\mathbb{R}^n} |\mathbb{B}|^q \, dx, \quad q = \operatorname{const} > 1, \quad \nu = \operatorname{const} > 0, \quad (8)$$

holds for all symmetric $(n \times n)$ matrices \mathbb{B} and all nonnegative g . Evidently, condition

$$\beta_0(g, \operatorname{tr} \mathbb{B}) \operatorname{tr} \mathbb{B} + \beta(g, |\mathbb{B}|) |\mathbb{B}|^2 \geq \nu |\mathbb{B}|^q \quad (8')$$

implies (8).

In the present work we are going to find a condition on the exponent q , the heat ratio γ and the dimension of space n such that smooth solutions to system (1), (2), (6)–(8) considered as laws of conservation of mass and momentum in the whole space do not exist globally in time.

The viscosity tensor \mathbb{P} of form (6) is the most widely used in the models of motion of incompressible non-Newtonian fluids, e.g. models due to Ostwald,

de Waele, Spriggs, Eyring, Carreau, Ellis, Bingham and many others ([2] and references therein). The model (6), (8') includes the power-law model as a special subclass. Power-law models are by far the most commonly used models for describing the response of a variety of fluids. Many colloids and suspensions are very well described by such models ([3],[4]), it is also a well accepted fact that blood, in a homogenized sense, can be modeled as a power-law fluid (see for example [5], [6]). The power-law models are quite popular in glaciology [7] and geology.

For compressible fluids most results concern with Newtonian fluid (local in time existence of classical solutions, global existence of weak solutions [8], [9]). The modern state of art in the mathematical theory of non-Newtonian fluid is described in [10], where homogeneous incompressible fluids are studied in sufficient details, while compressible fluids are hardly considered (only very weak measure-valued solutions are obtained). For stronger classes, the solvability of (1), (2) is proved for an arbitrary n (for bounded domain) in [11], [12], the paper [13] is devoted to the problem of improvement of the smoothness of the solution obtained, the global regularity estimates were derived in dimensions one and two. Thus, there are still many unsolved problems in the mathematical theory of non-Newtonian fluids, and first of all this is the problem of being globally well-posed.

Let us assume $\rho \geq 0$ and introduce the total mass

$$m = \int_{\mathbb{R}^n} \rho \, dx,$$

momentum

$$P = \int_{\mathbb{R}^n} \rho u \, dx$$

and total energy

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \left(\frac{1}{2} \rho |u|^2 + \frac{A \rho^\gamma}{\gamma - 1} \right) dx = E_k(t) + E_i(t),$$

where $E_k(t)$, and $E_i(t)$ are the kinetic and internal components of energy, respectively. The Hölder inequality implies that for a given solution the momentum is finite provided the total mass and total energy are finite. To obtain the finiteness of total mass and energy we have to specify the decay rate of density and velocity as $|x| \rightarrow \infty$.

Namely, let us consider initial data

$$\rho_0(x) \in L^1 \cap L^\gamma(\mathbb{R}^n), \quad \rho_0(x) \geq 0, \quad u_0(x) \in H_1^q(\mathbb{R}^n). \quad (9)$$

Definition 1 *We say that a solution (ρ, u) to the problem (1), (2), (7), (8),*

(9) belongs to the class \mathbb{K}_q if

$$\rho(t, x) \in C^1(0, T; \mathbb{L}^1 \cap \mathbb{L}^\gamma(\mathbb{R}^n)), \quad u(t, x) \in C^2(0, T; \mathbb{H}_1^q(\mathbb{R}^n)), \quad T > 0.$$

Proposition 1 *If $n \geq 2$ and $q \in [q_0, n)$, $q_0 = \frac{2n\gamma}{n(\gamma-1)+2\gamma}$, then the total mass and total energy are finite on the solutions of the class \mathbb{K}_q .*

Proof. It is evident that the total mass and the internal energy are finite. To prove the convergence of the integral of kinetic energy we need the following inequality:

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{qn}{n-q}} dx \right)^{\frac{n-q}{n}} \leq K \int_{\mathbb{R}^n} |Du|^q dx, \quad (10)$$

where the constant $K = \frac{q(n-1)}{2(n-q)} > 0$. The latter inequality holds for $u \in H_1^q(\mathbb{R}^n)$, $q \in (1, n)$, ([14], p.22) and follows from the Sobolev embedding. Further, let us note that if $\rho \in C^1(0, T; \mathbb{L}^1 \cap \mathbb{L}^\gamma(\mathbb{R}^n))$, then

$$\rho \in C^1(0, T; \mathbb{L}^\sigma(\mathbb{R}^n)), \quad \sigma \in (1, \gamma).$$

Indeed, the Hölder inequality implies

$$\int_{\mathbb{R}^n} \rho^\sigma dx \leq \left(\int_{\mathbb{R}^n} \rho dx \right)^{\frac{\gamma-\sigma}{\gamma-1}} \left(\int_{\mathbb{R}^n} \rho^\gamma dx \right)^{\frac{\sigma-1}{\gamma-1}}. \quad (11)$$

Thus, from (10) and (11) we obtain

$$2E_k(t) \leq \|\rho\|_{\mathbb{L}^\sigma(\mathbb{R}^n)} \|u\|_{\mathbb{L}^{\frac{2\sigma}{\sigma-1}}(\mathbb{R}^n)}^2 \leq \|\rho\|_{\mathbb{L}^\sigma(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |u|^{\frac{q_\sigma n}{n-q_\sigma}} dx \right)^{\frac{\sigma-1}{\sigma}} \leq$$

$$\text{const} \cdot \|\rho\|_{\mathbb{L}^\sigma(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |Du|^{q_\sigma} dx \right)^{\frac{(\sigma-1)n}{(n-q_\sigma)\sigma}},$$

where $q_\sigma = \frac{2n\sigma}{(\sigma-1)n+2\sigma}$. It is easy to check that q_σ decreases with σ and $1 < q_\sigma < n$. This completes the proof. \square

Remark 1 *As follows from Proposition 1, the total mass and energy are finite for the solutions from the class \mathbb{K}_q , therefore the total momentum is finite as well. However, to guarantee a natural property of conservation of total mass and momentum we should require additionally that ρ, u, Du vanish as $|x| \rightarrow \infty$ sufficiently fast at any fixed t . This allows to eliminate the surface integral in the Stokes formula. In terms of the Sobolev spaces it signifies that we consider ρ, u, Du from the space $\mathbb{H}_l^2(\mathbb{R}^n)$, $l > \frac{n}{2} + 1$ (e.g. [15], Sec.5.13). Then due to a sufficiently quick decay of ρ, u, Du as $|x| \rightarrow \infty$ and properties of the viscosity tensor \mathbb{P} the values of total mass m and total momentum P are conserved.*

Nevertheless, more natural way is to require the conservation of m and P in advance. Then the fact of conservation of these values dictates a fast decay of variables at infinity.

Theorem 1 Assume $n \geq 2$ and $q \in (q_0, n)$. If the initial momentum $P \neq 0$, then there exists no global in time solution (ρ, u) to the Cauchy problem (1), (2), (6)–(8), (9) from the class \mathbb{K}_q with conserved total mass and momentum.

Proof. We use property (8) to obtain the following estimate of the total energy rate:

$$\mathcal{E}'(t) \leq -\nu \int_{\mathbb{R}^n} |Du|^q dx. \quad (12)$$

The Hölder inequality implies

$$|P| = \left| \int_{\mathbb{R}^n} \rho u dx \right| \leq \left(\int_{\mathbb{R}^n} \rho^{\frac{qn}{n(q-1)+q}} dx \right)^{\frac{n(q-1)+q}{qn}} \left(\int_{\mathbb{R}^n} |u|^{\frac{qn}{n-q}} dx \right)^{\frac{n-q}{qn}}. \quad (13)$$

Further, using the Jensen inequality we have

$$\left(\frac{1}{m} \int_{\mathbb{R}^n} \rho^{\frac{qn}{n(q-1)+q}} dx \right)^{\frac{(\gamma-1)(n(q-1)+q)}{n-q}} \leq \frac{\int_{\mathbb{R}^n} \rho^\gamma dx}{m} = \frac{(\gamma-1)E_i(t)}{mA} \quad (14)$$

for

$$\frac{(\gamma-1)(n(q-1)+q)}{n-q} \geq 1. \quad (15)$$

Thus, (13) and (14) give

$$|P| \leq K_1 (E_i(t))^{\frac{n-q}{qn(\gamma-1)}} \left(\int_{\mathbb{R}^n} |u|^{\frac{qn}{n-q}} dx \right)^{\frac{n-q}{qn}}, \quad (16)$$

with a positive constant K_1 that depends on γ, n, m .

Further, once more we use inequality (10). Namely, (12), (16), (10) imply

$$\begin{aligned} \mathcal{E}'(t) &\leq -\nu \frac{|P|^q}{K_1^q K} (E_i(t))^{-\frac{n-q}{qn(\gamma-1)}} \leq \\ &\leq -\nu \frac{|P|^q}{K_1^q K} (\mathcal{E}(0))^{-\frac{n-q}{qn(\gamma-1)}} = \text{const} < 0. \end{aligned}$$

The latter inequality contradicts to nonnegativity of the total energy of system. Inequality (16) can be rewritten as $\gamma \geq \frac{qn}{n(q-1)+q}$ or $q > q_1 = \frac{n\gamma}{(n+1)(\gamma-1)+1}$. One can see that $q_0 > q_1$. Thus, the proof is over. \square

Now we consider the flow of compressible non-Newtonian magnetic fluid in the space \mathbb{R}^3 . The governing system is a combination of the compressible equations of non-Newtonian fluid and Maxwell's equations of electromagnetism:

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0, \quad (17)$$

$$\partial_t(\rho u) + \operatorname{Div}_x(\rho u \otimes u) = (\operatorname{curl}_x H) \times H + \operatorname{Div}_x \mathbb{S}, \quad (18)$$

$$\partial_t H - \operatorname{curl}_x(u \times H) = -\operatorname{curl}_x(\eta \operatorname{curl}_x H), \quad \operatorname{div}_x H = 0, \quad (19)$$

where the stress tensor \mathbb{S} obeys the conditions (6), (7), (8) and $H = (H_1, H_2, H_3)$ denotes the magnetic field, η is a nonnegative constant. We restrict ourselves to the barotropic case.

Although the electric field E does not appear in the MHD system (17 – 19), it is indeed induced according to the relation $E = \eta \operatorname{curl}_x H - u \times H$ by the moving conductive flow in the magnetic field.

The results on the classical solvability of the magnetohydrodynamic equations are very scanty even in the case of Newtonian flow. Let us mention the local in time existence of solutions to the Cauchy problem for the density separated from zero [16]. What about global solvability, even for the one-dimensional case, the global existence of classical solutions to the full perfect MHD equations with large data remains unsolved (smooth global solutions near the constant state in one-dimensional case are investigated in [17]). The existence of global weak solutions in the Newtonian case was proved recently [18] (see also [19] for references).

Now we are going to prove an analog of Theorem 1 for non-Newtonian MHD equations.

The total energy in this case is

$$\mathcal{E} = \int_{\mathbb{R}^3} \left(\frac{1}{2} \rho |u|^2 + \frac{|H|^2}{2} + \frac{A \rho^\gamma}{\gamma - 1} \right) dx = E_k(t) + E_m(t) + E_i(t).$$

Thus, there arises a new component $E_m(t)$, the magnetic energy.

Definition 2 We define the class \mathbb{K}_q^H of solutions (ρ, u, H) to system (17 – 19), (7), (8) as

$$\rho(t, x) \in C^1(0, T; \mathbb{L}^1 \cap \mathbb{L}^\gamma(\mathbb{R}^3)), \quad u(t, x) \in C^2(0, T; \mathbb{H}_1^q(\mathbb{R}^3)),$$

$$H(t, x) \in C^2(0, T; \mathbb{L}^2(\mathbb{R}^3)), \quad T > 0.$$

Theorem 2 Let $q \in [\frac{6\gamma}{5\gamma-3}, 3)$. If the initial momentum P does not vanish, then there exists no global in time solution (ρ, u, H) to the Cauchy problem

(17)– (19), (6) – (8) with data

$$\rho_0(x) \in \mathbb{L}^1 \cap \mathbb{L}^\gamma(\mathbb{R}^3), \quad u_0(x) \in \mathbb{H}_1^q(\mathbb{R}^3), \quad H_0(x) \in \mathbb{L}^2(\mathbb{R}^3),$$

from the class \mathbb{K}_q^H with conserved total mass and momentum.

Proof. As follows from Proposition 1, for $q \in [\frac{6\gamma}{5\gamma-3}, 3)$ the kinetic energy is finite for the solution from \mathbb{K}_q^H , therefore total energy, total mass and total momentum P are finite as well. Moreover, the momentum P and mass m are assumed to be conserved.

Further, we can estimate the derivative of the total energy \mathcal{E} as

$$\mathcal{E}'(t) \leq -\eta \int_{\mathbb{R}^3} |\operatorname{curl}_x H|^2 dx - \nu \int_{\mathbb{R}^3} |Du|^q dx \leq 0, \quad (21)$$

therefore $\mathcal{E}(t) \leq \mathcal{E}(0)$. We can rewrite literally the proof of Theorem 1 to get the inequality $\mathcal{E}'(t) \leq -const < 0$, which implies the contradiction with nonnegativity of total energy. \square

Remark 2 *In fact, Theorems 1 and 2 are extensions of results of [20] and [21] proved for the Navier-Stokes ($n \geq 3$) and MGD ($n = 3$) equations. For the viscoelastic fluid it is possible to consider also $n = 2$, whereas for the Newtonian fluid the problem of global in time existence of smooth solutions with finite mass and energy is open in the two-dimensional space. For smooth initial data with a compact support the nonexistence of global in time classical solutions in the case of the Newtonian fluid follows from [22] for any n .*

References

- [1] J.Serrin, *Mathematical Principles of Classical Fluid Mechanics*, Handbuch für Physik, Bd.VIII, Berlin (1959).
- [2] J.Málek, J.Nečas, K.P.Rajagopal, Global analysis of the flows of fluids with pressure-dependent viscosities, Arch.Rational Mech. Anal. 165 (2002) 243–269.
- [3] W.R.Schowalter, *Mechanics of non-Newtonian Fluids*. Pergamon Press, 1978.
- [4] R.B.Bird, R.S.Amstrong, O. Hassager, *Dynamics of Polymer Liquids*, Jon Wiley & Sons, New York, 2nd edition, 1993.
- [5] Y.I.Cho, K.R.Kenney, Effects of the non-Newtonian viscosity of blood on hemodynamics of diseased arterial flows. Adv.Bioengineering 15 (1989), 147–148.
- [6] G.R.Cokelet, The rheology of human blood, In: *Biomechanics: Its function and Objectives*, Editors Y.C.Fung, N.Perrone & M.Aliker, Prentice-Hall, 1972, 63 – 103.

- [7] K.Hutter, *Theoretical Glaciology*, Dordrecht: D.Reidel, 1983.
- [8] S.N.Antontsev, A.V.Kazhikhov, V.N.Monakhov, Boundary problems of mechanics of inhomogeneous liquids, Novosibirsk, 1983.
- [9] P.-L. Lions, Mathematical topics in fluid mechanics, Vol.2, Clarendon Press, Oxdord, 1996.
- [10] J.Malek, J.Necas, M.Rokyta, M.Růžička, *Weak and measure-valued solutions to evolutionary PDEs* Chapman & Hall, London, 1996.
- [11] A.E.Mamontov, Global solvability of multidimensional Navier-Stokes equations of compressible nonlinearly viscous fluid. I", Siberian Math.J, 40 (1999), 408 – 420.
- [12] A.E.Mamontov, Global solvability of multidimensional Navier-Stokes equations of compressible nonlinearly viscous fluid. II", Siberian Math.J, 40 (1999), 635 – 649.
- [13] A.E.Mamontov, Global regularity estimates for multidimensional equations of compressible non-Newtonian fluids, Mathematical Notes, 68 (2000), 312–325.
- [14] E.Hebey, Sobolev spaces on Riemannian manifolds. Lecture Notes in Mathematics. Springer Berlin/ Heidelberg. Vol.1635, 1996.
- [15] R.D. Richtmyer, Principles of advanced mathematical physics, Vol.1, Springer New York/ Heiderberg/ Berlin, 1978.
- [16] A.I.Volpert, S.I.Khudiaev, On the Cauchy problem for composite systems of nonlinear equations, Mat.Sbornik 87 (1972), N4, 504 – 528.
- [17] S. Kawashima, M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, Proc. Japan Acad. Ser. A Math. Sci. 58 (1982) 384 - 387.
- [18] X. Hu, D. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, Comm. Math. Phys. 283 (2008), 255 - 284.
- [19] X. Hu, D. Wang, Compactness of weak solutions to the three-dimensional compressible magnetohydrodynamic equations, J. Differential Equations 245 (2008) 2176 - 2198.
- [20] O. Rozanova, Blow up of smooth solutions to the compressible Navier-Stokes equations with the data highly decreasing at infinity J.Diff.Equat., 245 (2008), 1762 – 1774.
- [21] O. Rozanova, Blow up of smooth solutions to the barotropic compressible magnetohydrodynamic equations with finite mass and energy, Proceedings of Symposia in Applied Mathematics 2009. Hyperbolic Problems: Theory, Numerics and Ap-plications. American Mathematical Society. Volume: 67 (2009), 911 – 918.
- [22] Z.P.Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, Comm.Pure Appl.Math. 51(1998), 229 – 240.